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On the derivatives of curvature of framed space curve and their time-updating scheme

Mayank Chadha *, Michael D. Todd

University of California, San Diego, 9500 Gilman Drive, La Jolla CA 92093-0085
United States of America

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ABSTRACT

This paper considers the curvature of framed space curves, their higher-order derivatives, variations, and co-rotational derivatives. We realize that parametrizing rotation tensor using the *Gibbs vector* is effective in deriving a closed-form formula that gives the general n-th order derivative of the curvature tensor as the summation of functions of the parametrizing quantity and its own derivatives. We use these results for formulating a linearized updating algorithm for curvature and its derivatives when the configuration of the curve acquires a small increment.
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1. Introduction

The Frenet–Serret and Bishop frames [1] are two popular intrinsic curve-framing techniques. Their limitations in representing practical problems and their uniqueness criterion made us propose a system-dependent *material frame* [2], useful for considering practical problems like computer graphics, path-estimation, shape-sensing [3], etc. With appropriate restrictions and constraints, any frame may be obtained from the general material frame (refer to Section 4 of [2]). In this paper, we focus our attention on the curvature of the material frame and its derivatives.

Among the numerous practical applications of framed space curves, extensive research in the area of geometrically-exact beams subjected to large deformation and finite strains have been done in past (see [4–7] and the references therein). Seminal contributions have been made tackling theoretical and computational techniques associated with modeling and solving for geometrically exact beams, e.g., problems related to interpolation and discretization techniques (see [8]), composite beams (see [9]) and variational principles (see [10,11]). We were presented the need to compute higher-order derivatives of curvature while investigating

* Corresponding author.
E-mail addresses: machadha@eng.ucsd.edu (M. Chadha), mdtodd@eng.ucsd.edu (M.D. Todd).

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our own higher-order, geometrically-exact beam/rod theory. The kinematics of beam/rods under arbitrarily-large deformations defined in Chadha and Todd [12] renders the deformation map not only to be a function of curvature, but also a function of its higher-order derivatives. Numerical solution of such problems (e.g., through finite difference or finite element analysis) requires updating these kinematic quantities. Secondly, in the problems concerning the path-estimation and dynamics of geometrically-exact beams, angular velocity and acceleration become important. Geometrically, angular velocity and the curvature tensor belong to same space in the theory of rotational Lie group. In such problems, derivatives of the curvature tensor (or angular velocity tensor) gain importance.

We conclude this introduction with a note on notation and definitions. The n dimensional Euclidean space is represented by $\mathbb{R}^n$. The space of real numbers and integers is denoted by $\mathbb{R}$ and $\mathbb{Z}$, with $\mathbb{R}^+$ and $\mathbb{Z}^+$ giving the set of positive real numbers and integers (including 0) respectively. The dot product, ordinary vector product and tensor product of two Euclidean vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are defined as $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$, $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, and $\boldsymbol{v}_1 \otimes \boldsymbol{v}_2$, respectively. The Euclidean norm is represented by $\|.\|$ or the un-bolded version of the symbol (for example, $\|\boldsymbol{v}\| = v$). The n-th order partial derivative with respect to a scalar quantity, ξ for instance, is given by the operator ∂_ξ^n. For $n, i \in \mathbb{Z}^+$ and $n \geq i \geq 0$, the binomial coefficient is defined as $C_i^n = \frac{n!}{i!(n-i)!}$.

2. Framed space curves

2.1. Finite rotation

A framed space curve, parameterized by the arc-length $\xi \in [0, L]$, is defined by the position vector $\boldsymbol{\varphi}(\xi) \in \mathbb{R}^3$ and the orthonormal material frame field $\{\boldsymbol{d}_i(\xi)\}$ calibrated with respect to fixed orthonormal reference frame $\{\boldsymbol{E}_i\}$ such that we may define the orthogonal rotation tensor $\boldsymbol{Q}(\xi)$ as:

$$\boldsymbol{d}_i(\xi) = \boldsymbol{Q}(\xi).\boldsymbol{E}_i; \quad \boldsymbol{Q}(\xi) = \sum_{i=1}^{3} \boldsymbol{d}_i(\xi) \otimes \boldsymbol{E}_i. \tag{1}$$

Finite rotations are represented by an element of a non-linear compact proper-orthogonal Lie group $SO(3)$ that has a linear skew-symmetric matrix as its Lie algebra $so(3)$, defined as

$$SO(3) := \{\boldsymbol{Q} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 |\ \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}_3, \text{ and } \det \boldsymbol{Q} = 1\}; \tag{2a}$$

$$so(3) := \{\hat{\boldsymbol{\theta}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 |\ \hat{\boldsymbol{\theta}} \text{ is linear, and } \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}^T = \boldsymbol{0}_3\}. \tag{2b}$$

In the equation above, $\boldsymbol{0}_3$ represents the 3×3 zero matrix, whereas $\boldsymbol{I}_3 = \sum_{i,j=1}^{3} \delta_{ij} \boldsymbol{E}_i \otimes \boldsymbol{E}_j$ is the identity tensor with respect to which the director frame field is calibrated. The anti-symmetric tensor $\hat{\boldsymbol{\theta}} \in so(3)$ is equivalent to the associated *axial vector* $\boldsymbol{\theta} \in \mathbb{R}^3$ in the sense that for any vector $\boldsymbol{v} \in \mathbb{R}^3$, we have $\hat{\boldsymbol{\theta}}.\boldsymbol{v} = \boldsymbol{\theta} \times \boldsymbol{v}$. Therefore, there exists an isomorphism between $\mathbb{R}^3$ and $so(3)$. From here on, any matrix quantity with a hat $(\hat{.})$ represents an anti-symmetric matrix. We define the Lie-bracket of two anti-symmetric matrices as $[.,.] : so(3) \times so(3) \longrightarrow \mathbb{R}^3$, such that for any $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}} \in so(3)$ with corresponding axial vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, respectively, and any vector $\boldsymbol{v} \in \mathbb{R}^3$, we have

$$\left[\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right] = (\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}} - \hat{\boldsymbol{b}}.\hat{\boldsymbol{a}}); \tag{3a}$$

$$\left[\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right].\boldsymbol{v} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{v}. \tag{3b}$$

The definition of $SO(3)$ in Eq. (2a) allows rotation tensor to be parameterized by a rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$ (with corresponding anti-symmetric matrix $\hat{\boldsymbol{\theta}} \in so(3)$ and the norm $\|\boldsymbol{\theta}\| = \theta$). This is attained by means of exponential map $\exp : so(3) \longrightarrow SO(3)$ such that

$$\boldsymbol{Q}(\boldsymbol{\theta}) = \boldsymbol{I}_3 + \frac{\sin\theta}{\theta}\ \hat{\boldsymbol{\theta}} + \frac{(1-\cos\theta)}{\theta^2}\hat{\boldsymbol{\theta}}^2 = \sum_{i\geq 0} \frac{\hat{\boldsymbol{\theta}}^i}{i!} = \exp(\hat{\boldsymbol{\theta}}). \tag{4}$$

2.2. Curvature: Material and spatial form

Curvature tensor defines local change of the triad. We have

$$\partial_\xi \boldsymbol{d}_i = \partial_\xi \boldsymbol{Q}.\boldsymbol{E}_i = \partial_\xi \boldsymbol{Q}.\boldsymbol{Q}^T.\boldsymbol{d}_i = \hat{\boldsymbol{\kappa}}.\boldsymbol{d}_i. \tag{5}$$

Here, $\hat{\boldsymbol{\kappa}} = \partial_\xi \boldsymbol{Q}.\boldsymbol{Q}^T$ gives the curvature tensor. It is an anti-symmetric matrix with the corresponding axial vector $\boldsymbol{\kappa} = \overline{\kappa}_i \boldsymbol{d}_i$, known as the curvature vector. We define $T_{\boldsymbol{Q}}SO(3)$ as the tangent plane of a non-linear $SO(3)$ manifold, such that $\partial_\xi \boldsymbol{Q} = \hat{\boldsymbol{\kappa}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$. If rotation tensor $\boldsymbol{Q}$ is parameterized by rotation vector $\boldsymbol{\theta}$, using Eqs. (1) and (5), we obtain the following expression of the curvature tensor $\hat{\boldsymbol{\kappa}}$ as

$$\hat{\boldsymbol{\kappa}} = \partial_\xi \exp(\hat{\boldsymbol{\theta}}).\exp(-\hat{\boldsymbol{\theta}}) = \left(\frac{\sin\theta}{\theta}\right)\partial_\xi\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\theta}{\theta^2}\right)\left[\hat{\boldsymbol{\theta}}, \partial_\xi\hat{\boldsymbol{\theta}}\right] + (\boldsymbol{\theta}\cdot\partial_\xi\boldsymbol{\theta})\left(\frac{\theta-\sin\theta}{\theta^3}\right)\hat{\boldsymbol{\theta}}. \tag{6}$$

From the expression above and the definition of Lie brackets in Eq. (3), the relationship between the curvature vector $\boldsymbol{\kappa}$ and the vector $\partial_\xi\boldsymbol{\theta}$ can be deduced as

$$\boldsymbol{\kappa} = \overbrace{\left(\left(\frac{\sin\theta}{\theta}\right)\boldsymbol{I}_3 + \left(\frac{1-\cos\theta}{\theta^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\theta-\sin\theta}{\theta^3}\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta}\right)}^{\boldsymbol{T}_{\boldsymbol{\theta}}}.\partial_\xi\boldsymbol{\theta} = \boldsymbol{T}_{\boldsymbol{\theta}}.\partial_\xi\boldsymbol{\theta}. \tag{7}$$

We define the material curvature tensor as $\hat{\overline{\boldsymbol{\kappa}}} = \boldsymbol{Q}^T.\hat{\boldsymbol{\kappa}}.\boldsymbol{Q} = \boldsymbol{Q}^T.\partial_\xi\boldsymbol{Q} \in so(3)$ obtained by parallel transport of $\hat{\boldsymbol{\kappa}}.\boldsymbol{Q}$ from $T_{\boldsymbol{Q}}SO(3) \longrightarrow so(3)$. From Eq. (1), $\boldsymbol{Q}$ represents the finite rotation, while $\hat{\overline{\boldsymbol{\kappa}}}$ gives an infinitesimal rotation with respect to the calibrating frame (fixed reference frame) $\{\boldsymbol{E}_i\}$, and $\boldsymbol{Q}.\hat{\overline{\boldsymbol{\kappa}}} = \hat{\boldsymbol{\kappa}}.\boldsymbol{Q}$ represents an infinitesimal rotation with respect to the $\{\boldsymbol{d}_i\}$ frame. In the physical context of rotation, the tangent vector $\boldsymbol{Q}.\hat{\overline{\boldsymbol{\kappa}}}$ and $\boldsymbol{\kappa}.\boldsymbol{Q}$ performs an infinitesimal rotation with respect to $\{\boldsymbol{d}_i\}$ frame, but the quantity $\boldsymbol{Q}.\hat{\overline{\boldsymbol{\kappa}}}$ is obtained by a left translation of $\hat{\overline{\boldsymbol{\kappa}}} \in so(3)$ to $\boldsymbol{Q}.\hat{\overline{\boldsymbol{\kappa}}} \in T_{\boldsymbol{Q}}SO(3)$. Moreover, $\hat{\boldsymbol{\kappa}}.\boldsymbol{Q}$ represents the superimposition of infinitesimal rotation contributed by $\hat{\boldsymbol{\kappa}}$ onto the finite rotation due to $\boldsymbol{Q}$ (this is also called the right translation of $\hat{\boldsymbol{\kappa}} \in so(3)$ to the tangent vector $\hat{\boldsymbol{\kappa}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$). The former kind of tangent vector fields are known as *left-invariant* and the latter as *right-invariant* fields.

2.3. Material and spatial quantities and their derivatives

As we have defined the material and spatial forms of the curvature vector (and tensor), it is rather useful to define a vector $\boldsymbol{v} \in \mathbb{R}^3$ in its material and spatial form. Consider spatial and material representations of a vector, $\boldsymbol{v} = \overline{v}_i\boldsymbol{d}_i$ and $\overline{\boldsymbol{v}} = \overline{v}_i\boldsymbol{E}_i$ respectively, such that $\boldsymbol{v} = \boldsymbol{Q}.\overline{\boldsymbol{v}}$. The derivative of these vectors is obtained as

$$\begin{gathered}\partial_\xi\boldsymbol{v} = \partial_\xi\overline{v}_i.\boldsymbol{d}_i + \overline{v}_i.\partial_\xi\boldsymbol{d}_i = \tilde{\partial}_\xi\boldsymbol{v} + \boldsymbol{\kappa}\times\boldsymbol{v};\\ \partial_\xi\overline{\boldsymbol{v}} = \partial_\xi\overline{v}_i.\boldsymbol{E}_i = \boldsymbol{Q}^T.\tilde{\partial}_\xi\boldsymbol{v}.\end{gathered} \tag{8}$$

In the equation above, $\tilde{\partial}_\xi\boldsymbol{v}$ defines the co-rotational derivative of spatial vector $\boldsymbol{v}$. It essentially gives the change in components of the vector $\boldsymbol{v}$, provided the frame of reference is assumed to be fixed. Geometrically, it is obtained by parallel-transport (left translation) of the vector $\partial_\xi\overline{\boldsymbol{v}}$.

Along similar lines, consider the spatial and material representations of the tensor $\boldsymbol{A} = \sum_{i,j=1}^3 \overline{A}_{ij}\boldsymbol{d}_i\otimes\boldsymbol{d}_j$ and $\overline{\boldsymbol{A}} = \sum_{i,j=1}^3 \overline{A}_{ij}\boldsymbol{E}_i\otimes\boldsymbol{E}_j$ respectively, such that $\boldsymbol{A} = \boldsymbol{Q}.\overline{\boldsymbol{A}}.\boldsymbol{Q}^T$. Realizing $\partial_\xi\boldsymbol{Q} = \hat{\boldsymbol{\kappa}}.\boldsymbol{Q}$ and $\partial_\xi\boldsymbol{Q}^T = -\boldsymbol{Q}^T.\hat{\boldsymbol{\kappa}}$, we have the following

$$\partial_\xi\boldsymbol{A} = \boldsymbol{Q}.\partial_\xi\overline{\boldsymbol{A}}.\boldsymbol{Q}^T + \partial_\xi\boldsymbol{Q}.\overline{\boldsymbol{A}}.\boldsymbol{Q}^T + \boldsymbol{Q}.\overline{\boldsymbol{A}}.\partial_\xi\boldsymbol{Q}^T = \tilde{\partial}_\xi\boldsymbol{A} + \hat{\boldsymbol{\kappa}}.\boldsymbol{A} - \boldsymbol{A}.\hat{\boldsymbol{\kappa}}. \tag{9}$$

In the equation above, we define the co-rotational derivative of the tensor $\boldsymbol{A}$ as $\tilde{\partial}_\xi\boldsymbol{A} = \boldsymbol{Q}.\partial_\xi\overline{\boldsymbol{A}}.\boldsymbol{Q}^T$. Physically, it gives the change in components of tensor $\boldsymbol{A}$ setting the reference frame constant. From here on, $\tilde{\partial}_x(.)$ represents the co-rotational derivative of any quantity $(.)$ with respect to the variable x. We now present some propositions describing higher order co-rotational derivatives.

Proposition 1. *For any vector $\boldsymbol{v} \in \mathbb{R}^3$, define the operator $\hat{\partial}_\xi$ such that $\hat{\partial}_\xi^n \boldsymbol{v} = \hat{\boldsymbol{\kappa}}^n.\boldsymbol{v}$, where, for example $\hat{\boldsymbol{\kappa}}^3 = \hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}}$. The n-th order co-rotational derivative $\tilde{\partial}_\xi^n$ is then given by $(\partial_\xi - \hat{\partial}_\xi)^n$ such that the order of operations are not commutative (for example, $\partial_\xi \hat{\partial}_\xi \neq \hat{\partial}_\xi \partial_\xi$) and $\tilde{\partial}_\xi^0 = (\partial_\xi - \hat{\partial}_\xi)^0$ is an identity operator.*

Proof. We prove Proposition 1 by using the principle of mathematical induction. Consider a vector $\boldsymbol{v} \in \mathbb{R}^3$, assuming that $\tilde{\partial}_\xi^n \boldsymbol{v} = (\partial_\xi - \hat{\partial}_\xi)^n \boldsymbol{v}$, for $n = 1$ and 2, we have

$$\begin{aligned} &\tilde{\partial}_\xi^1 \equiv \tilde{\partial}_\xi = (\partial_\xi - \hat{\partial}_\xi); \\ &\tilde{\partial}_\xi^2 = (\partial_\xi - \hat{\partial}_\xi)^2 = \partial_\xi^2 - \partial_\xi \hat{\partial}_\xi - \hat{\partial}_\xi \partial_\xi + \hat{\partial}_\xi^2. \end{aligned} \tag{10}$$

such that

$$\tilde{\partial}_\xi \boldsymbol{v} = \partial_\xi \boldsymbol{v} - \hat{\partial}_\xi \boldsymbol{v} = \partial_\xi \boldsymbol{v} - \hat{\boldsymbol{\kappa}}.\boldsymbol{v}; \tag{11a}$$

$$\tilde{\partial}_\xi^2 \boldsymbol{v} = \partial_\xi^2 \boldsymbol{v} - \partial_\xi \hat{\partial}_\xi \boldsymbol{v} - \hat{\partial}_\xi \partial_\xi \boldsymbol{v} + \hat{\partial}_\xi^2 \boldsymbol{v} = \partial_\xi^2 \boldsymbol{v} + (\hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}} - \partial_\xi \hat{\boldsymbol{\kappa}}).\boldsymbol{v} - 2\hat{\boldsymbol{\kappa}}.\partial_\xi \boldsymbol{v}. \tag{11b}$$

We now prove that the equation set (11) may be derived using the definition of co-rotational derivative in Eq. (8). Eq. (11a) is true by definition (refer to Eq. (8)). Taking the derivative of Eq. (11a) yields

$$\partial_\xi \tilde{\partial}_\xi \boldsymbol{v} = \partial_\xi^2 \boldsymbol{v} - \partial_\xi \hat{\boldsymbol{\kappa}}.\boldsymbol{v} - \hat{\boldsymbol{\kappa}}.\partial_\xi \boldsymbol{v} \tag{12}$$

We note from the definition of co-rotational derivative in Eq. (8) that

$$\partial_\xi \tilde{\partial}_\xi \boldsymbol{v} = \tilde{\partial}_\xi^2 \boldsymbol{v} + \hat{\boldsymbol{\kappa}}.\tilde{\partial}_\xi \boldsymbol{v} = \tilde{\partial}_\xi^2 \boldsymbol{v} + \hat{\boldsymbol{\kappa}}.(\partial_\xi \boldsymbol{v} - \hat{\boldsymbol{\kappa}}.\boldsymbol{v}) = \tilde{\partial}_\xi^2 \boldsymbol{v} + \hat{\boldsymbol{\kappa}}.\partial_\xi \boldsymbol{v} - \hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}}.\boldsymbol{v}. \tag{13}$$

From Eqs. (12) and (13), we have

$$\tilde{\partial}_\xi^2 \boldsymbol{v} = \partial_\xi \tilde{\partial}_\xi \boldsymbol{v} - \hat{\boldsymbol{\kappa}}.\partial_\xi \boldsymbol{v} + \hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}}.\boldsymbol{v} = \partial_\xi^2 \boldsymbol{v} + (\hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{\kappa}} - \partial_\xi \hat{\boldsymbol{\kappa}}).\boldsymbol{v} - 2\hat{\boldsymbol{\kappa}}.\partial_\xi \boldsymbol{v}. \tag{14}$$

Expression obtained in (14) is same as (11b). We can continue the process explained above and realize that for any n, $\tilde{\partial}_\xi^n = (\partial_\xi - \hat{\partial}_\xi)(\partial_\xi - \hat{\partial}_\xi)^{(n-1)} = (\partial_\xi - \hat{\partial}_\xi)^n$. This completes the proof. □

Proposition 2. *For any $\hat{\boldsymbol{A}} \in so(3)$ with the corresponding axial vector $\boldsymbol{A} \in \mathbb{R}^3$, the recurrence formula for the n-th order co-rotational derivative $\tilde{\partial}_\xi^n \hat{\boldsymbol{A}} \in so(3)$ is given as*

$$\tilde{\partial}_\xi^n \hat{\boldsymbol{A}} = \partial_\xi^n \hat{\boldsymbol{A}} - (1 - \delta_{n0}) \sum_{i=1}^{n} \partial_\xi^{(i-1)} \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-i)} \hat{\boldsymbol{A}}\right]. \tag{15}$$

Proof. From the definition of co-rotational derivatives and Lie-bracket in Eq. (9) and (3a) respectively, we have

$$\tilde{\partial}_\xi \hat{\boldsymbol{A}} = \partial_\xi \hat{\boldsymbol{A}} - \hat{\boldsymbol{\kappa}}.\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}.\hat{\boldsymbol{\kappa}} = \partial_\xi \hat{\boldsymbol{A}} - [\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{A}}] \tag{16}$$

Since $\partial_\xi^m \hat{\boldsymbol{A}} \in so(3)$ for any $m \in \mathbb{Z}^+$, the result above can be used to obtain the recurrence-relation for nth order co-rotational derivative. For $n \in \mathbb{Z}^+ - \{0\}$, we have

$$\begin{aligned} \tilde{\partial}_\xi^n \hat{\boldsymbol{A}} =& \partial_\xi \left(\tilde{\partial}_\xi^{(n-1)} \hat{\boldsymbol{A}}\right) - \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-1)} \hat{\boldsymbol{A}}\right] = \partial_\xi \left(\partial_\xi \left(\tilde{\partial}_\xi^{(n-2)} \hat{\boldsymbol{A}}\right) - \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-2)} \hat{\boldsymbol{A}}\right]\right) - \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-1)} \hat{\boldsymbol{A}}\right] \\ =& \partial_\xi^n \hat{\boldsymbol{A}} - \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-1)} \hat{\boldsymbol{A}}\right] - \partial_\xi \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-2)} \hat{\boldsymbol{A}}\right] - \cdots - \partial_\xi^{(n-1)} \left[\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{A}}\right] = \partial_\xi^n \hat{\boldsymbol{A}} - \sum_{i=1}^{n} \partial_\xi^{(i-1)} \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-i)} \hat{\boldsymbol{A}}\right]. \end{aligned} \tag{17}$$

We note that for $n = 0$ we have $\tilde{\partial}_\xi^0 \hat{\boldsymbol{A}} = \partial_\xi^0 \hat{\boldsymbol{A}} = \hat{\boldsymbol{A}}$. Thus, the sum part in the equation above vanish for $n = 0$, justifying the use of $(1 - \delta_{n0})$ factor in Eq. (15). This completes the proof. □

Proposition 2 can be extended for any tensor $\boldsymbol{B}$ as

$$\tilde{\partial}_\xi^n \boldsymbol{B} = \partial_\xi^n \boldsymbol{B} - (1-\delta_{n0}) \sum_{i=1}^{n} \partial_\xi^{(i-1)} \left(\hat{\boldsymbol{\kappa}}.\tilde{\partial}_\xi^{(n-i)} \boldsymbol{B} - \tilde{\partial}_\xi^{(n-i)} \boldsymbol{B}.\hat{\boldsymbol{\kappa}} \right) \tag{18}$$

Proposition 3. *For spatial vector and tensor* $\boldsymbol{v}$ *and* $\boldsymbol{B}$ *respectively, with corresponding material quantities* $\overline{\boldsymbol{v}}$ *and* $\overline{\boldsymbol{B}}$*, the n-th order co-rotational derivative* $\tilde{\partial}_\xi^n \boldsymbol{v}$ *and* $\tilde{\partial}_\xi^n \boldsymbol{B}$ *can be obtained by left-translation of the n-th order derivative of the respective material quantities such that*

$$\tilde{\partial}_\xi^n \boldsymbol{v} = \boldsymbol{Q}.\partial_\xi^n \overline{\boldsymbol{v}} \tag{19a}$$

$$\tilde{\partial}_\xi^n \boldsymbol{B} = \boldsymbol{Q}.\partial_\xi^n \overline{\boldsymbol{B}}.\boldsymbol{Q}^T. \tag{19b}$$

Proof. This proposition can be easily proved using product rule on $\overline{\boldsymbol{v}} = \boldsymbol{Q}^T.\boldsymbol{v}$ and $\overline{\boldsymbol{B}} = \boldsymbol{Q}^T.\boldsymbol{B}.\boldsymbol{Q}$ and substituting for $\partial_\xi \boldsymbol{Q}^T = -\boldsymbol{Q}^T.\hat{\boldsymbol{\kappa}}$. The result obtained after such computations (say for $n = 1, 2, 3$) when compared with the results obtained Proposition 1 and Eq. (18), proves the intended result. □

2.4. Variation and linearization of the rotation tensor

To obtain the virtual rotation tensor field, we superimpose an admissible variation field $\delta\boldsymbol{Q}$ on the rotation field $\boldsymbol{Q}$. The varied configuration is then defined by $\boldsymbol{Q}_\epsilon$ such that for $\epsilon > 0$, we have $\boldsymbol{Q}_\epsilon = \boldsymbol{Q}(\boldsymbol{\theta} + \epsilon\delta\boldsymbol{\theta}) = \boldsymbol{Q}(\epsilon\delta\boldsymbol{\alpha}).\boldsymbol{Q}(\boldsymbol{\theta})$. We also note that it is advantageous, in regard to developing a curvature updating scheme in *Eulerian* sense (refer to Section 4), to express the virtual rotation tensor by means of *virtual rotation vector in current state* $\delta\boldsymbol{\alpha}$ (that is saying $\delta\hat{\boldsymbol{\alpha}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$) contrary to the *variation of total rotation vector* $\delta\boldsymbol{\theta}$ ($\delta\hat{\boldsymbol{\theta}} \in so(3)$). We arrive at the expression of varied rotation tensor as:

$$\delta\boldsymbol{Q} = \partial_\epsilon \boldsymbol{Q}_\epsilon|_{\epsilon=0} = \partial_\epsilon \left(\exp(\epsilon\delta\hat{\boldsymbol{\alpha}}).\exp(\hat{\boldsymbol{\theta}}) \right)|_{\epsilon=0} = \left(\delta\hat{\boldsymbol{\alpha}}.\exp(\epsilon\delta\hat{\boldsymbol{\alpha}}).\exp(\hat{\boldsymbol{\theta}}) \right)|_{\epsilon=0} = \delta\hat{\boldsymbol{\alpha}}.\boldsymbol{Q}(\boldsymbol{\theta}). \tag{20}$$

Here, $\delta\hat{\boldsymbol{\alpha}}$ represents the anti-symmetric matrix associated with the vector $\delta\boldsymbol{\alpha}$. We note that Proposition 3 also holds for the variation and mix of derivatives and variation. Like the variation, we define the linearized part of rotation tensor $\boldsymbol{Q}$, linearized at $\exp(\hat{\boldsymbol{\theta}})$ in the direction of $\Delta\hat{\boldsymbol{\alpha}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$ as

$$\Delta\boldsymbol{Q} = \partial_\epsilon \boldsymbol{Q}_\epsilon|_{\epsilon=0} \text{ with } \boldsymbol{Q}_\epsilon = \boldsymbol{Q}(\epsilon\Delta\boldsymbol{\alpha}).\boldsymbol{Q}(\boldsymbol{\theta}). \tag{21}$$

We observe that $\hat{\boldsymbol{\kappa}}$ and $\Delta\hat{\boldsymbol{\alpha}}$ belongs to same tangent space in the sense that $\partial_\xi \boldsymbol{Q} = \hat{\boldsymbol{\kappa}}.\boldsymbol{Q}, \Delta\boldsymbol{Q} = \Delta\hat{\boldsymbol{\alpha}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$, whereas $\partial_\xi\hat{\boldsymbol{\theta}}, \Delta\hat{\boldsymbol{\theta}} \in so(3)$. This observation along with the result in Eqs. (6) and (7) yields the relationship between the vector $\Delta\hat{\boldsymbol{\theta}}$ (or $\Delta\boldsymbol{\theta}$) and $\Delta\hat{\boldsymbol{\alpha}}$ (or $\Delta\boldsymbol{\alpha}$) as

$$\Delta\hat{\boldsymbol{\alpha}} = \left(\frac{\sin\theta}{\theta}\right)\Delta\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\theta}{\theta^2}\right)\left[\hat{\boldsymbol{\theta}}, \Delta\hat{\boldsymbol{\theta}}\right] + (\boldsymbol{\theta}\cdot\Delta\hat{\boldsymbol{\theta}})\left(\frac{\theta-\sin\theta}{\theta^3}\right)\hat{\boldsymbol{\theta}}; \tag{22a}$$

$$\Delta\boldsymbol{\alpha} = \boldsymbol{T}_{\boldsymbol{\theta}}.\Delta\boldsymbol{\theta}. \tag{22b}$$

3. On derivatives

3.1. Useful results on derivatives of Lie-bracket and higher-order product rule

Proposition 4. *For any* $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}} \in so(3)$ *with corresponding axial vectors* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ *respectively, the following formula for derivative holds:*

$$\partial_\xi^n \left[\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right] = \sum_{i=0}^{n} C_i^n \left[\partial_\xi^{(n-i)}\hat{\boldsymbol{a}}, \partial_\xi^{(i)}\hat{\boldsymbol{b}}\right] = \sum_{i=0}^{n} C_i^n \left[\partial_\xi^{(i)}\hat{\boldsymbol{a}}, \partial_\xi^{(n-i)}\hat{\boldsymbol{b}}\right] \tag{23}$$

Proof. Using the definition of Lie-bracket in Eq. (3a), we have

$$\partial_\xi \left[\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right] = (\partial_\xi \hat{\boldsymbol{a}}.\hat{\boldsymbol{b}} + \hat{\boldsymbol{a}}.\partial_\xi \hat{\boldsymbol{b}}) - (\partial_\xi \hat{\boldsymbol{b}}.\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}.\partial_\xi \hat{\boldsymbol{a}}) = \left[\partial_\xi \hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right] + \left[\hat{\boldsymbol{a}}, \partial_\xi \hat{\boldsymbol{b}}\right]. \tag{24}$$

Higher-order derivatives of the Lie-bracket derived using the above result yields an expression given by Eq. (23). The first and the second equality in (23) hold since $C_i^n = C_{(n-i)}^n$. This completes the proof. □

The result above can be extended to obtain general product rule; from Eq. (24), we see that Lie-brackets follow the chain rule just like the product of two scalars or a dot product except for the fact that Lie-brackets are non-commutative. For instance, for scalar functions $f(\xi)$ and a tensor $\boldsymbol{A}(\xi)$, we have

$$\partial_\xi^n (f\boldsymbol{A}) = \sum_{i=0}^{n} C_i^n \partial_\xi^{(n-i)} f.\partial_\xi^{(i)} \boldsymbol{A} = \sum_{i=0}^{n} C_i^n \partial_\xi^{(i)} f.\partial_\xi^{(n-i)} \boldsymbol{A}. \tag{25}$$

Proposition 5. *For any $\hat{\boldsymbol{a}}(\xi) \in so(3)$, the following holds:*

$$\partial_\xi^m \left[\hat{\boldsymbol{a}}, \partial_\xi \hat{\boldsymbol{a}}\right] = \sum_{j=0}^{j_m^{max}} b_{mj} \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(m-j+1)} \hat{\boldsymbol{a}}\right]. \tag{26}$$

where $m, j, j_m^{max}, b_{mj} \in \mathbb{Z}^+$, such that the coefficients j_m^{max}, and b_{mj} are given as

$$j_m^{max} = \begin{cases} \mathit{floor}\left(\frac{m+1}{2}\right), & \mathit{if}\ \frac{m+1}{2} \notin \mathbb{Z}^+ \\ \frac{m+1}{2} - 1, & \mathit{if}\ \frac{m+1}{2} \in \mathbb{Z}^+ \end{cases}; \ \mathit{and}\ b_{mj} = C_j^m - C_{(m-j+1)}^m = C_j^m \left(\frac{m-2j+1}{m-j+1}\right). \tag{27}$$

Proof. Using the result (23) of Proposition 4, we get,

$$\partial_\xi^m [\hat{\boldsymbol{a}}, \partial_\xi \hat{\boldsymbol{a}}] = \sum_{j=0}^{m} C_j^m \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(m-j+1)} \hat{\boldsymbol{a}}\right] = \overbrace{C_0^m \left[\hat{\boldsymbol{a}}, \partial_\xi^{(m+1)} \hat{\boldsymbol{a}}\right]}^{\text{Term 0}} + \overbrace{\sum_{j=1}^{m} C_j^m \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(m-j+1)} \hat{\boldsymbol{a}}\right]}^{\text{Term 1}}. \tag{28}$$

The terms in the expansion of *Term 1* vanish when $j = \frac{m+1}{2}$. Keeping that in mind, we note that the expansion of *Term 1* can be written in two possible ways: the first possibility is when $j > \frac{m+1}{2}$, and the second option is considering $j < \frac{m+1}{2}$. In the first option, the coefficient of all the terms in the sum will be negative, whereas, for second case, the coefficients will be positive. This owes to the anti-commutative property of Lie-brackets. We consider the second case in our derivations.

We can further simplify *Term 1*. The total number of terms present in the expanded form of *Term 1* is less than m. This is because the terms with interchanged order of derivatives in Lie-bracket can be reduced to one term. For instance:

$$c_1 \left[\partial_\xi^x \hat{\boldsymbol{a}}, \partial_\xi^y \hat{\boldsymbol{a}}\right] + c_2 \left[\partial_\xi^y \hat{\boldsymbol{a}}, \partial_\xi^x \hat{\boldsymbol{a}}\right] = (c_1 - c_2) \left[\partial_\xi^x \hat{\boldsymbol{a}}, \partial_\xi^y \hat{\boldsymbol{a}}\right].$$

Thus, the maximum value of j is restricted by the fact that $j < \frac{m+1}{2}$ and $j \in \mathbb{Z}^+ - \{0\}$. Combining these two constraints yields $\max(j) = j_m^{\max}$ given by Eq. (27). However, such a reduction or simplification of *Term 1* changes the coefficient by which each term in the sum is weighed. The discussion presented so far may be demonstrated, for example, for $m = 4$ as:

$$\begin{aligned} \text{Term 1}|_{m=4} &= \sum_{j=1}^{4} C_j^4 \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(5-j)} \hat{\boldsymbol{a}}\right] = 3\left[\partial_\xi \hat{\boldsymbol{a}}, \partial_\xi^4 \hat{\boldsymbol{a}}\right] + 2\left[\partial_\xi^2 \hat{\boldsymbol{a}}, \partial_\xi^3 \hat{\boldsymbol{a}}\right] = -3\left[\partial_\xi^4 \hat{\boldsymbol{a}}, \partial_\xi \hat{\boldsymbol{a}}\right] - 2\left[\partial_\xi^3 \hat{\boldsymbol{a}}, \partial_\xi^2 \hat{\boldsymbol{a}}\right] \\ &= \|C_1^4 - C_4^4\| \left[\partial_\xi \hat{\boldsymbol{a}}, \partial_\xi^4 \hat{\boldsymbol{a}}\right] + \|C_2^4 - C_3^4\| \left[\partial_\xi^2 \hat{\boldsymbol{a}}, \partial_\xi^3 \hat{\boldsymbol{a}}\right] = \sum_{j=1}^{j_4^{\max}=2} (C_j^4 - C_{(4-j+1)}^4) \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(5-j)} \hat{\boldsymbol{a}}\right]. \end{aligned} \tag{29}$$

For a general case, if $j_m^{\max} \geq 1$, *Term 1* can be written as:

$$\text{Term } 1 = \sum_{j=1}^{j_m^{\max}} b_{mj} \left[\partial_\xi^{(j)} \hat{\boldsymbol{a}}, \partial_\xi^{(m-j+1)} \hat{\boldsymbol{a}} \right]. \tag{30}$$

The modified coefficient $b_{mj} = (C_j^m - C_{(m-j+1)}^m)$ is defined in Eq. (27). From the second equality in Eq. (27), we also note that $b_{m0} = C_0^m = 1$. Therefore, combining *Term 0* and *Term 1* proves the proposition. □

3.2. Derivatives of curvature tensor

The derivative of the curvature tensor may be obtained using Eq. (6) deploying a straightforward application of the chain rule. However, deriving the expression of higher-order derivatives using Eq. (6) is cumbersome because of the involvement of trigonometric functions. Instead, we realize that the reparametrization of the rotation tensor by the *Gibbs vector* (the components of which are called as *Gibbs* or *Rodriguez parameters* in the literature) yields the formula of curvature tensor that is beneficial in obtaining the derivative of curvature tensor in the form of a single summation-formula. Consider a rotation tensor $\boldsymbol{Q}(\boldsymbol{\theta}) = \exp(\hat{\boldsymbol{\theta}}) \in SO(3)$. We define the Gibbs vector $\boldsymbol{\phi}$ and the associated quantities as:

$$\hat{\boldsymbol{\phi}} = \frac{\tan\left(\frac{\theta}{2}\right)}{\theta} \hat{\boldsymbol{\theta}}; \quad \boldsymbol{\phi} = \frac{\tan\left(\frac{\theta}{2}\right)}{\theta} \boldsymbol{\theta}; \quad \phi = \|\boldsymbol{\phi}\| = \tan\left(\frac{\theta}{2}\right); \quad \overline{\phi} = 2\cos^2\left(\frac{\theta}{2}\right) = \frac{2}{\phi^2 + 1}. \tag{31}$$

The result defined in Eq. (4) may be manipulated using the definition given above as:

$$\boldsymbol{Q}(\boldsymbol{\phi}) = \boldsymbol{I}_3 + \overline{\phi}(\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2). \tag{32}$$

With this reparametrized form of rotation tensor, the curvature tensor can then be obtained in a much simpler form as:

$$\hat{\boldsymbol{\kappa}} = \partial_\xi \boldsymbol{Q}.\boldsymbol{Q}^T = \overline{\phi}\left(\partial_\xi \hat{\boldsymbol{\phi}} + [\hat{\boldsymbol{\phi}}, \partial_\xi \hat{\boldsymbol{\phi}}]\right). \tag{33}$$

Proposition 6. *The n-th order derivative of curvature tensor is given by:*

$$\partial_\xi^n \hat{\boldsymbol{\kappa}} = \sum_{i=0}^{n} C_i^n \partial_\xi^{(i)} \overline{\phi} \left(\partial_\xi^{(n-i+1)} \hat{\boldsymbol{\phi}} + \sum_{j=0}^{j_{(n-i)}^{max}} b_{(n-i)j} \left[\partial_\xi^{(j)} \hat{\boldsymbol{\phi}}, \partial_\xi^{(n-i+1-j)} \hat{\boldsymbol{\phi}} \right] \right) \tag{34}$$

where, $n, i, j, j_{(n-i)}^{max}, C_i^n, b_{(n-i)j} \in \mathbb{Z}^+$. Replacing $m \longrightarrow (n-i)$ in (27) yields $j_{(n-i)}^{max}$, and $b_{(n-i)j}$.

Proof. We utilize Eq. (25) and the expression of curvature tensor in Eq. (33) and obtain

$$\partial_\xi^n \hat{\boldsymbol{\kappa}} = \sum_{i=0}^{n} C_i^n \partial_\xi^{(i)} \overline{\phi}. \left(\partial_\xi^{(n-i)} \left(\partial_\xi \hat{\boldsymbol{\phi}} + \left[\hat{\boldsymbol{\phi}}, \partial_\xi \hat{\boldsymbol{\phi}} \right] \right) \right) = \sum_{i=0}^{n} C_i^n \partial_\xi^{(i)} \overline{\phi}. \left(\partial_\xi^{(n-i+1)} \hat{\boldsymbol{\phi}} + \partial_\xi^{(n-i)} \left[\hat{\boldsymbol{\phi}}, \partial_\xi \hat{\boldsymbol{\phi}} \right] \right) \tag{35}$$

Using Proposition 5 and replacing $m \longrightarrow (n-i)$, we obtain the expression of $\partial_\xi^{(n-i)} \left[\hat{\boldsymbol{\phi}}, \partial_\xi \hat{\boldsymbol{\phi}} \right]$, which when substituted in (35) arrives at the intended result. □

Corollary 1. *The following holds:*

$$\tilde{\partial}_\xi^n \hat{\boldsymbol{\kappa}} = \partial_\xi^n \hat{\boldsymbol{\kappa}} - (1 - \delta_{n0}) \sum_{i=1}^{n-1} \partial_\xi^{(i-1)} \left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-i)} \hat{\boldsymbol{\kappa}} \right] = \partial_\xi^n \hat{\boldsymbol{\kappa}} - (1 - \delta_{n0}) \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} C_j^{(i-1)} \left[\partial_\xi^{(j)} \hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-i-j)} \tilde{\partial}_\xi^{(n-i)} \hat{\boldsymbol{\kappa}} \right]; \tag{36a}$$

$$\partial_\xi^n \hat{\tilde{\boldsymbol{\kappa}}} = \boldsymbol{Q}^T . \tilde{\partial}_\xi^n \hat{\boldsymbol{\kappa}} . \boldsymbol{Q} \tag{36b}$$

Proof. This corollary follows from Propositions 2–4. We also note that in the sums presented above, $\max(i) = (n-1)$, because $\left[\hat{\boldsymbol{\kappa}}, \tilde{\partial}_\xi^{(n-i)}\hat{\boldsymbol{\kappa}}\right]\Big|_{(n=i)} = \mathbf{0}_3$. □

4. Updating the curvature and its derivatives

In this section, we shall address the situation where the space curve is evolving in discrete time, such that the transformed curve is also parameterized spatially by ξ. At time t, let the *initial* rotation tensor field be $\boldsymbol{Q}(\xi,t) = \boldsymbol{Q}_{\mathrm{i}}(\xi) \in SO(3)$ and in the next time step $(t+1)$, the *updated* (or *final*) rotation tensor field is $\boldsymbol{Q}(\xi,t+1) = \boldsymbol{Q}_{\mathrm{f}}(\xi) \in SO(3)$. We assume *Eulerian updating* of rotation tensor field, i.e. the change in rotation tensor field from discrete time step t to $(t+1)$ is given by an *incremental current rotation vector field* $\Delta\boldsymbol{\alpha}$, such that $\Delta\hat{\boldsymbol{\alpha}}.\boldsymbol{Q}_{\mathrm{i}} \in T_{\boldsymbol{Q}_{\mathrm{i}}}SO(3)$. We are given the derivative fields $\partial_\xi^n\Delta\boldsymbol{\alpha}$ (or $\partial_\xi^n\Delta\hat{\boldsymbol{\alpha}}$) and $\partial_\xi^n\boldsymbol{Q}_{\mathrm{i}}$ up to order n. The question posed is thus: "How do we obtain the updated curvature tensor, its spatial, material and co-rotational derivatives up to order $(n-1)$?". To proceed, we first present the updated rotation tensor as

$$\boldsymbol{Q}_{\mathrm{f}} = \exp(\Delta\hat{\boldsymbol{\alpha}}).\boldsymbol{Q}_{\mathrm{i}} = \boldsymbol{Q}_+.\boldsymbol{Q}_{\mathrm{i}} \text{ where, } \boldsymbol{Q}_+ = \exp(\Delta\hat{\boldsymbol{\alpha}}). \tag{37}$$

We define the curvature corresponding to *incremental current rotation vector* $\Delta\boldsymbol{\alpha}$ and the *transport operator* $\mathbb{T}_{\boldsymbol{Q}}$ as:

$$\begin{aligned}
\hat{\boldsymbol{\kappa}}_+ &= \partial_\xi\exp(\Delta\hat{\boldsymbol{\alpha}}).\exp(-\Delta\hat{\boldsymbol{\alpha}}) = \partial_\xi\boldsymbol{Q}_+.\boldsymbol{Q}_+^T;\\
\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}] &= \boldsymbol{Q}.\hat{\boldsymbol{A}}.\boldsymbol{Q}^T \in so(3), \quad \forall \quad \boldsymbol{Q} \in SO(3), \hat{\boldsymbol{A}} \in so(3).
\end{aligned} \tag{38}$$

Proposition 7. *The nth order derivative of the transport operator* $\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]$ *is given by*

$$\partial_\xi^n\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}] = \mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^n\hat{\boldsymbol{A}}] + (1-\delta_{n0})\sum_{k=1}^{n}\sum_{i=0}^{n-k} C_i^{(n-k)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-k-i)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^{(k-1)}\hat{\boldsymbol{A}}]\right]. \tag{39}$$

Proof. Consider

$$\partial_\xi\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}] = \boldsymbol{Q}.\partial_\xi\hat{\boldsymbol{A}}.\boldsymbol{Q}^T + \hat{\boldsymbol{\kappa}}.\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}] - \mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}].\hat{\boldsymbol{\kappa}} = \mathbb{T}_{\boldsymbol{Q}}[\partial_\xi\hat{\boldsymbol{A}}] + [\hat{\boldsymbol{\kappa}}, \mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]]. \tag{40}$$

Using the above result along with Proposition 4, for $n \geq 1$, we have

$$\begin{aligned}
\partial_\xi^n\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}] =& \partial_\xi^{(n-1)}.\left(\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi\hat{\boldsymbol{A}}] + [\hat{\boldsymbol{\kappa}}, \mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]]\right) = \partial_\xi^{(n-1)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi\hat{\boldsymbol{A}}] + \sum_{i=0}^{(n-1)} C_i^{(n-1)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-1-i)}\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]\right]\\
=& \partial_\xi^{(n-2)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^2\hat{\boldsymbol{A}}] + \sum_{i=0}^{(n-2)} C_i^{(n-2)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-2-i)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi\hat{\boldsymbol{A}}]\right] + \sum_{i=0}^{(n-1)} C_i^{(n-1)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-1-i)}\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]\right]\\
=& \mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^n\hat{\boldsymbol{A}}] + \left(\sum_{i=0}^{(n-1)} C_i^{(n-1)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-1-i)}\mathbb{T}_{\boldsymbol{Q}}[\hat{\boldsymbol{A}}]\right] + \sum_{i=0}^{(n-2)} C_i^{(n-2)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-2-i)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi\hat{\boldsymbol{A}}]\right] + \cdots\right.\\
&\left. + \sum_{i=0}^{0} C_i^0\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-i)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^{(n-1)}\hat{\boldsymbol{A}}]\right]\right) = \mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^n\hat{\boldsymbol{A}}] + \sum_{k=1}^{n}\sum_{i=0}^{n-k} C_i^{(n-k)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}, \partial_\xi^{(n-k-i)}\mathbb{T}_{\boldsymbol{Q}}[\partial_\xi^{(k-1)}\hat{\boldsymbol{A}}]\right].
\end{aligned} \tag{41}$$

Since the sum in Eq. (39) vanishes at $n = 0$, the factor $(1-\delta_{n0})$ is justified. □

Proposition 8. *Let* $\hat{\boldsymbol{\kappa}}_i = \partial_\xi\boldsymbol{Q}_i.\boldsymbol{Q}_i^T$ *and* $\hat{\boldsymbol{\kappa}}_f = \partial_\xi\boldsymbol{Q}_f.\boldsymbol{Q}_f^T$ *denote the curvature field corresponding to the initial and final configurations respectively. The updated curvature tensor and its derivatives are given by the recurrence-formula*

$$\partial_\xi^n\hat{\boldsymbol{\kappa}}_f = \partial_\xi^n\hat{\boldsymbol{\kappa}}_+ + \mathbb{T}_{\boldsymbol{Q}_+}[\partial_\xi^n\hat{\boldsymbol{\kappa}}_i] + (1-\delta_{n0})\sum_{k=1}^{n}\sum_{i=0}^{n-k} C_i^{(n-k)}\left[\partial_\xi^{(i)}\hat{\boldsymbol{\kappa}}_+, \partial_\xi^{(n-k-i)}\mathbb{T}_{\boldsymbol{Q}_+}[\partial_\xi^{(k-1)}\hat{\boldsymbol{\kappa}}_i]\right]. \tag{42}$$

Proof. Using Eqs. (37) and (38), we obtain updated curvature as:

$$\hat{\boldsymbol{\kappa}}_{\mathrm{f}} = \partial_\xi(\boldsymbol{Q}_+.\boldsymbol{Q}_i).(\boldsymbol{Q}_+.\boldsymbol{Q}_i)^T = \partial_\xi(\boldsymbol{Q}_+).\boldsymbol{Q}_+^T + \boldsymbol{Q}_+.\hat{\boldsymbol{\kappa}}_{\mathrm{i}}.\boldsymbol{Q}_+^T = \hat{\boldsymbol{\kappa}}_+ + \mathbb{T}_{\boldsymbol{Q}_+}[\hat{\boldsymbol{\kappa}}_{\mathrm{i}}]. \tag{43}$$

Substituting Eq. (39) obtained in Proposition 7 in $\partial_\xi^n\hat{\boldsymbol{\kappa}}_{\mathrm{f}} = \partial_\xi^n\hat{\boldsymbol{\kappa}}_+ + \partial_\xi^n\mathbb{T}_{\boldsymbol{Q}_+}[\hat{\boldsymbol{\kappa}}_{\mathrm{i}}]$ proves this proposition. □

The curvatures $\hat{\boldsymbol{\kappa}}_+$ and their derivatives may be obtained using Proposition 6. Once the spatial curvature and its derivatives are obtained, the material derivative and co-rotational derivative may be obtained using Proposition 1 and Corollary 1.

5. Conclusion

In this paper, we primarily focused our attention on the curvature tensor and its derivatives associated with any space curve framed by a general *material frame*. In addition to discussing the spatial and material forms of the curvature tensor, we have investigated the higher-order derivatives and co-rotational derivatives of these quantities. We have presented, for the first time to our knowledge, a closed-form formula for all higher-order derivative of the spatial curvature tensor. Finally, a time-updating algorithm for curvature (both spatial and material) and its derivatives (partial and co-rotational) was presented, which is particularly useful in practical problems like finite element formulation of geometrically-exact beams among many other applications.

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